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Applications of non-classical symmetries

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Abstract. In this paper we discuss new applications of the non-classical method of symmetry reduction. The equations treated are well known in mathematical physics and are used to describe multisoliton solutions of the Burger equation. They are also useful in the study of Kac–Moody algebras, and in the examination of nonlinear waves in the theory of elasticity. With the help of the non-classical symmetry method the partial differential equations are reduced to ordinary differential equations which are solvable explicitly.

1. Introduction

At the end of the last century Sophus Lie developed a method for reducing the order of ordinary differential equations (ODEs) and for finding new solutions for ordinary and partial differential equations (PDEs) [1, 2]. This method determines the transformation under which a differential equation is invariant. Since the calculations are very complicated and extensive, the use of this method was ignored for a long time.

In 1969 Bluman and Cole [18] proposed an extension of Lie's method for symmetry reduction which has since been called the 'non-classical method' for symmetry reduction, or in short 'the non-classical method'. With this method Bluman and Cole found new solutions of the heat equation which are not derivable by Lie's 'classical method'. Olver and Rosenau extended the non-classical method to 'weak symmetries' [19]. Levi and Winternitz [20] presented an explicit algorithm to calculate the determining equations of the non-classical method and found classes of solutions for the Boussinesq equation not derivable with Lie's symmetry reduction. Quite recently Clarkson and Mansfield [21, 22] proposed another algorithm for the non-classical method that can be applied even to a wider class of PDEs. The main difference between the Levi–Winternitz algorithm and the Clarkson–Mansfield algorithm is in the use of the differential consequences resulting from the conditional equation and their substitution into prolonged equations. While these consequences are substituted by Levi and Winternitz in the prolonged expression of the original PDE, Clarkson and Mansfield use them to substitute the original PDE.

In this paper we show, using three examples which are well known and have been extensively studied in various fields of physics, that the non-classical method is an excellent tool for finding new classes of solutions. The desired solutions of the PDEs cannot be found by applying local symmetry methods. The reduction of the PDE to an ordinary differential equation is demonstrated, and can be solved analytically by well known methods or numerically for specific initial values.

In section 2 we summarize the essentials of Lie's symmetry reduction. Section 3 contains the main ingredients of the non-classical method. Sections 4–6 are devoted to discussing the applications of the non-classical method. In section 4, we consider an evolution equation for solitons describing a single soliton for an N-soliton solution of the Burger equation. Section 5 contains results for a coupled system of KdV equations used in the study of Kac–Moody algebras. In section 6, the case of the nonlinear wave equation for a moving threadline is treated. The last section discusses our results and gives a summary of the method.

2. The classical method of Lie symmetry reduction

A common way of finding particular solutions for a system of nonlinear PDEs is the symmetry reduction by Sophus Lie [1, 2]. This method is entirely algorithmic and allows one to calculate the symmetry group represented by infinitesimal transformations under which solutions of the system are invariant. This method has been applied to hundreds of PDEs and systems of PDEs in order to obtain exact similarity solutions [5–7]. An extensive number of other equations treated by the same method are compiled in the books of Roger and Ames [9] as well as in Ibragimov's survey [8].

The Lie method is also a useful tool for finding exact solutions, constructing new solutions from old ones and characterizing the symmetry properties of PDEs. In particular, we are interested in discovering solutions for well known systems of PDEs.

Everyone who has used Lie's method when studying solutions of PDEs knows that this procedure is very time consuming and tedious. Especially, the large number of determining equations resulting from this procedure is hard to handle by a pencil calculation. This was one of the reasons for developing computer-based methods to do the calculations. Various symbolic manipulating programs have been developed in recent years. Today there exists a program in nearly every algebraic language such as MATHEMATICA [12], REDUCE [15], MAPLE [16], MACSYMA [13] and AXIOM [15]. A more recent summary of these programs and their functionality is given in [14]. Some of these programs [12, 15, 17] are now capable of deriving completely automatically the symmetries and their algebraic properties of a great number of equations. With our program in MATHEMATICA we solved nearly 300 equations found in the literature. Parts of the results obtained are listed in [8, 9] which we used to check our program.

To understand how Lie's method works and what information we can gain from it, let us discuss its general procedure. We consider the general case of a nonlinear system of PDEs which we describe by

$$\Delta_\nu(x, u^{(k)}) = 0 \qquad \nu = 1, \ldots, m \tag{1}$$

where Δ_ν represents one of the m coupled equations in n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. $u^{(k)}$ denotes all derivatives of the dependent variables u with respect to the independent variables x up to order k. We assume that the Δ_ν are smooth functions in their arguments. The central question of a symmetry analysis of (1) is under which transformation such a system is invariant. The invariance can be considered following Lie by applying the one-parameter Lie group of point-transformations to (1):

$$(x^i)^* = x^i + \epsilon\xi^i(x, u) + \mathrm{O}(\epsilon^2) \tag{2}$$

$$(u^\alpha)^* = u^\alpha + \epsilon\phi^\alpha(x, u) + \mathrm{O}(\epsilon^2)\,. \tag{3}$$

This means Lie's method requires the invariance of (1) under the transformation (2), (3). Claiming the invariance of (1) yields an overdetermined *linear* system of PDEs for the infinitesimals ξ^i and ϕ^α. The transformations of the independent and dependent variables are characterized by vector fields or infinitesimal generators given by

$$X = \sum_{i=1}^{n} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{m} \phi^\alpha(x, u)\partial_{u^\alpha} . \tag{4}$$

The mathematical formulation of the invariance criterion for (1) is

$$X^{(k)}\ \Delta_\nu(x, u^{(k)})\big|_{\Delta=0} = 0 \tag{5}$$

where $X^{(k)}$ denotes the kth prolongation of the infinitesimal generator X,

$$X^{(k)} = X + \sum_{\alpha=0}^{m} \sum_{J} \phi_J^\alpha \partial_{u_J^\alpha} \tag{6}$$

with the multi-indices $J = (j_1, \ldots, j_p)$ and

$$u_J^\alpha = \frac{\partial^p u^\alpha}{\partial x^{j_1} \ldots \partial x^{j_p}} .$$

The second summation is extended over all derivatives of u^α up to order k. The higher prolongation elements are determined by the infinitesimals ξ^i and ϕ^α through

$$\phi_J^\alpha = D_J(\phi^\alpha(x, u) - \sum_{i=1}^{n} \xi^i(x, u)u_i^\alpha) + \sum_{i=1}^{n} \xi^i(x, u)u_{J,i}^\alpha . \tag{7}$$

$D_i F(x, u)$ is the ith total derivative of a function $F(x, u)$ which is defined by

$$D_i F(x, u) = \partial_{x^i} F(x, u) + \sum_{\alpha=1}^{m} u_i^\alpha \partial_{u^\alpha} F(x, u) .$$

The ϕ_J^α can be calculated recursively by a formula known from [3, 4]

$$\phi_{J,l}^\alpha = D_l \phi_J^\alpha - \sum_{i=1}^{n} u_{J,i}^\alpha D_l \xi^i . \tag{8}$$

Once we know the infinitesimals ξ^i and ϕ^α we also know the infinitesimal generator X. With the infinitesimal generator at hand the explicit transformation acting on the space of independent and dependent variables can be calculated by the so-called Lie series

$$(x^*, u^*) = \exp[\epsilon X]\,(x, u) . \tag{9}$$

Knowing the explicit transformations in (9), new solutions can be obtained from old ones. This behaviour was already known by Lie in the study of the diffusion equation.

Another method to discover new solutions of (1) is to construct a class of functions which solve (1) and are invariant under a subgroup of the full symmetry group of (1). The group invariants can be calculated by solving the characteristic equations

$$\frac{\mathrm{d}x^1}{\xi^1(x, u)} = \cdots = \frac{\mathrm{d}x^n}{\xi^n(x, u)} = \frac{\mathrm{d}u^1}{\phi^1(x, u)} = \cdots = \frac{\mathrm{d}u^m}{\phi^m(x, u)} \tag{10}$$

or the invariant surface condition

$$\sum_{i=1}^{n} \xi^i(x, u)u_{x^i}^\alpha - \phi^\alpha(x, u) = 0 . \tag{11}$$

Then the original system (1) can be rewritten in terms of group invariants and thus the number of independent variables is reduced.

Based on the invariant surface condition (11), in 1969 Bluman and Cole developed an extension of Lie's procedure which is today known as the non-classical method of symmetry reduction.

3. Non-classical method of symmetry reduction

The 'non-classical method' was introduced by Bluman and Cole [18] during the study of the diffusion equation. A group-theoretical explanation of this method is given in [19, 20]. The procedure was extended in [19] to 'weak symmetries' and 'side conditions'. The vector fields of the non-classical method do not need to form a Lie algebra. Hence there can be a wider class of similarity solutions than in the classical case. However, it occurs many times that the results of both the classical and non-classical methods are equivalent. In such a case the non-classical method fails to produce new solutions.

An essential observation by Bluman and Cole was that an invariant solution $u(x)$ of (1) does not only solve the original system but also the surface condition

$$X(u) = 0.$$

The non-classical method applies the classical Lie method of symmetry reduction to the extended system

$$\Delta_\nu(x, u) = 0 \qquad (12)$$

$$X(u^\alpha) = 0 \qquad (13)$$

where $\nu = 1, \ldots, m$, $\alpha = 1, \ldots, m$.
The invariance criterion reads in the original form as

$$X^{(k)}\ \Delta(x, u)|_{\Delta=0, X(u)=0} = 0 \qquad (14)$$

$$X^{(1)}\ X(u)|_{\Delta=0, X(u)=0} = 0. \qquad (15)$$

Equation (15) imposes no additional conditions because (15) is satisfied identically.

The added surface condition can be interpreted as a side condition or as a conditional equation that introduces new dependencies on the derivatives of u. These relations have to be taken into consideration during further calculations. In the algorithm of Levi–Winternitz the differential consequences of (13) are substituted in (14) to eliminate the additional relation among the derivatives of u. On the other hand, in the algorithm proposed by Clarkson and Mansfield the substitutions from the added surface condition (13) are replaced in the original system (12). Then the classical symmetry reduction by Lie is calculated for the altered system of PDEs.

Common to both algorithms is that the resulting determining equations for the infinitesimals ξ^i and ϕ^α are an overdetermined *nonlinear* system of PDEs. Because of the additional relations among the derivatives the set of solutions may, but need not be, larger than the one obtained by the classical method. These solutions yield neither a vector space nor a Lie algebra. With the non-classical method, the old solutions of (1) cannot be transformed to new ones, but similarity solutions can be obtained.

The non-classical method has been applied to various PDEs. New classes of solutions which cannot be obtained by the classical method have been found for the heat equation [18], the Boussinesq equation [20], the Burger equation [29] and the Fizhugh–Nagumo equation [28].

Since the calculation is very time-consuming, we have developed a MATHEMATICA program which allows the derivation of the *nonlinear* determining equations for non-classical symmetries. This alteration of Lie's method is incorporated in the Lie package [12], which also allows the solution of the nonlinear determining equations for some simple cases.

To demonstrate how we can find unknown solutions of well known equations we will apply the classical and non-classical procedure to some examples.

4. Evolution equation of solitons

The first example we will consider is an equation presented by Fuchssteiner [23]

$$SS_t - SS_{xx} + 2S_x^2 - 2mSS_x = 0\,. \tag{16}$$

This equation arises in the study of multi-soliton solutions of the Burger equation

$$u_t = u_{xx} + 2uu_x\,.$$

Equation (16) describes the evolution of a single soliton interacting with other solitons of the multi-soliton solution. An N-soliton solution of the Burger equation is composed as a superposition of N solutions of equation (16) with different 'masses' m_i. The masses m_i are related in the corresponding N-soliton solution to the asymptotic speed of the emerging solitons. For a detailed discussion of that equation and the link to similar equations refer to [23].

If we apply the classical method of Lie described in section 2, we can find the infinitesimals which represent the point symmetries of (16). The infinitesimals contain, besides the equation parameter m, six arbitrary constants and one free function:

$$\xi^1 = c_2 + c_3 t - \frac{c_3}{2m}x + \frac{c_5}{m}x + 4c_6 tx \tag{17}$$

$$\xi^2 = c_1 - \frac{c_3}{m}t + \frac{2c_5}{m}t + 4c_6 t^2 \tag{18}$$

$$\phi^1 = S(c_4 + 2c_6 t + 2c_5 mt + 4c_6 m^2 t^2 + c_5 x + 4c_6 mtx + c_6 x^2) + S^2 f(x,t)\,. \tag{19}$$

The free function $f(x,t)$ has to solve the linearized form of the original equation (16)

$$-f_t + 2mf_x + f_{xx} = 0\,. \tag{20}$$

ξ^1 is related to the x and ξ^2 to the t coordinates while ϕ^1 is the infinitesimal for the dependent variable $S(x,t)$.

The symmetry analysis by Lie yields a discrete six-dimensional symmetry group for (16). The transformations characterized by the infinitesimals corresponding to the group constants c_1, c_2 and c_4 reflect the invariance of (16), under space and time translations, respectively, the homogeneity of the original equation. Additionally there exist two different kinds of scaling invariant solutions which can be obtained by setting $c_5 = 1$, respectively $c_6 = 1$, and the rest of the group constants c_i to zero. The invariant solution corresponding to c_5 is

$$S(x,t) = w(z)\exp\left[mx + m^2 t\right]$$

with the scaling invariant similarity variable

$$z = \frac{x}{\sqrt{t}}\,.$$

The ansatz connected with c_6 has a similar form, but with additional terms

$$S(x,t) = w(z)\frac{1}{\sqrt{t}}\exp\left[m^2 t + mx + \frac{x^2}{4t}\right] \qquad \text{with} \quad z = \frac{x}{t}\,.$$

Finally, there is another type of invariant solution which is derived by setting $c_3 = 1$ and the rest to 0. This solution is not scaling invariant and has a rather simple form

$$S(x,t) = w(z) \qquad \text{with} \quad z = \frac{x}{\sqrt{t}} + 2m\sqrt{t}\,.$$

The intention here is to demonstrate that the application of the non-classical method will give us new solutions not obtainable with the classical method. We show that the solutions of the non-classical method contain a wider class of functions than the classical method, and that the results of the classical method are included in the non-classical one. To demonstrate how we can apply the non-classical method to (16) let us first consider the standard case with $\xi^2 = 1$, usually discussed in the literature [20–22].

Case 1: $\xi^2 = 1$. If we apply the theory given in section 3 with $\xi^2 = 1$ to (16) we find that the infinitesimals have the general form

$$\xi^1 = a(x,t) \tag{21}$$

$$\xi^2 = 1 \tag{22}$$

$$\phi^1 = b(x,t)S + c(x,t)S^2 . \tag{23}$$

Inserting this ansatz into the determining equations we obtain a *nonlinear* coupled system of PDEs for the unknown functions a, b and c depending only on x and t:

$$-a_t - 2ma_x - 2aa_x + 2b_x + a_{xx} = 0 \tag{24}$$

$$b_t + 2ba_x - 2mb_x - b_{xx} = 0 \tag{25}$$

$$c_t + 2ca_x - 2mc_x - c_{xx} = 0 . \tag{26}$$

If we set $a_x = 0$ or either $a_x =$ constant in (24)–(26) we get results equivalent to those found by the classical method. Since system (24)–(26) is a nonlinear system of PDEs, it is hard to find a complete and general solution. However, a particular solution which is different from the solutions obtainable from the classical infinitesimals follows by setting $b(x,t) = c(x,t) = 0$ and $a(x,t) = f(x)$:

$$\xi^1 = f(x) \qquad \xi^2 = 1 \qquad \phi^1 = 0 .$$

Now integrating (24) gives an ODE of first order for $f(x)$

$$f'(x) - 2mf(x) - f^2(x) = k_1 . \tag{27}$$

Via this equation $f(x)$ can be found explicitly by

$$f(x) = -m + \sqrt{k_1 - m^2}\tan\left(\sqrt{k_1 - m^2}(x + k_2)\right) . \tag{28}$$

The surface condition following from (11) gives us

$$f(x)S_x + S_t = 0 . \tag{29}$$

Consequently, $S(x,t)$ has the form

$$S(x,t) = w(z) \qquad z = \mathrm{e}^{-t}\exp\left[\int^x \frac{\mathrm{d}s}{f(s)}\right] \tag{30}$$

with

$$z = k\exp\left[-t - \frac{m}{k_1}x + \frac{1}{k_1}\log\left|-m\cos\sqrt{k_1 - m^2}(x + k_2) + \sqrt{k_1 - m^2}\sin\sqrt{k_1 - m^2}(x + k_2)\right|\right].$$

The original equation (16) is reduced by this 'ansatz' to an ODE

$$-zww'' + (k_1 - 1)ww' + 2zw'^2 = 0 \tag{31}$$

which solution reads

$$w(z) = \frac{k_2}{-1 + k_3 z^{k_1}} .$$

The integration constant k of z can be replaced by k_3. This type of solution containing a new kind of similarity variable, where transcendental functions appear instead of powers in x and t, is completely different from solutions obtainable from the classical method. For another case let us now consider that $\xi^2 = 0$.

Case 2: $\xi^2 = 0$. For this case, we use an ansatz for the infinitesimals of

$$\xi^1 = 1 \qquad \xi^2 = 0 \qquad \phi^1 = \phi^1(x, t, S).$$

With this ansatz, we obtain the determining equation for ϕ^1,

$$-2(\phi^1)^3 + 2S(\phi^1)^2\phi^1_S - S^2(\phi^1)^2\phi^1_{SS} + S^2\phi^1_t - 2mS^2\phi^1_x + 4S\phi^1\phi^1_x \\ -2S^2\phi^1\phi^1_{Sx} - S^2\phi^1_{xx} = 0. \qquad (32)$$

The use of a polynomial ansatz in S for ϕ^1 delivers a solution quadratic in S

$$\phi^1 = g(x,t)S(x,t) + f(x,t)S^2(x,t). \qquad (33)$$

The free functions $f(x,t)$ and $g(x,t)$ again have to solve a coupled *nonlinear* system of differential equations

$$f_t - 2mf_x + 2fg_x - f_{xx} = 0 \qquad (34)$$

$$g_t - 2mg_x + 2gg_x - g_{xx} = 0. \qquad (35)$$

As a limiting case, we get the classical result with $c_1 = \cdots = c_3 = 0,\ c_4 = 1,\ c_5 = c_6 = 0$ from the non-classical one by setting $g(x,t) = 1$ because (34) reduces to (20).

It is easy to recognize two solutions of the system (34)–(35) which are not equivalent to classical results. To see how this is possible let us consider two cases for the choice of f and g.

(i) $f(x,t) = g(x,t) = f_1(x)$. The substitution of the ansatz into (34) and (35) reduces them to a single second-order ODE for f_1 which can be integrated at once:

$$-f_1' - 2mf_1 + f_1^2 = k_1. \qquad (36)$$

This equation can be solved in terms of tanh:

$$f_1(x) = m - \sqrt{k_1 + m^2}\tanh\left[\sqrt{k_1 + m^2}(x + k_2)\right]. \qquad (37)$$

The surface condition yields for $S(x,t)$

$$S(x,t) = -\frac{1}{F(t)\exp\left[-\int f_1(x)\right] + 1} \qquad (38)$$

with

$$\int f_1(x)\,\mathrm{d}x = mx - \log\left|\cosh\sqrt{k_1 + m^2}(x + k_2)\right|.$$

Re-substituting (38) into the original equation (16) determines $F(t)$ to be

$$F(t) = k_3 e^{k_1 t}.$$

A graphical representation of $S(x,t)$ for a specific choice of the parameters is plotted in figure 1.

The ansatz (38) is a special kind of separation in x and t. In turn, a non-classical method can indicate whether a kind of separation is possible and determines the structure of the ansatz. In this case there is no similarity variable at all, and consequently this solution cannot be derived by the classical method.

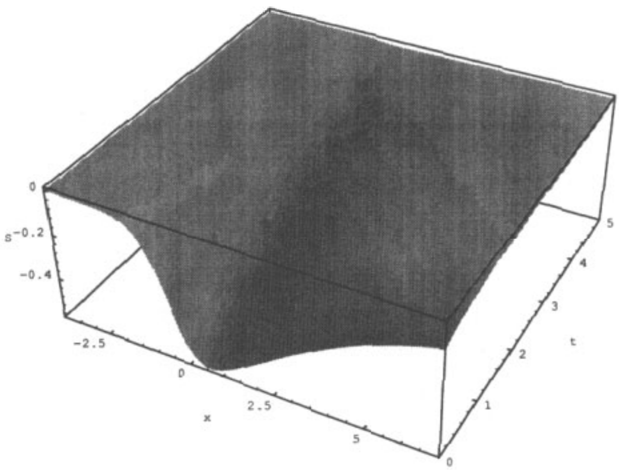

Figure 1. Solution of equation (16) for the case 2 (i). The choice of the parameters to plot the function is $k_2 = 0, k_1 = k_3 = m = 1$.

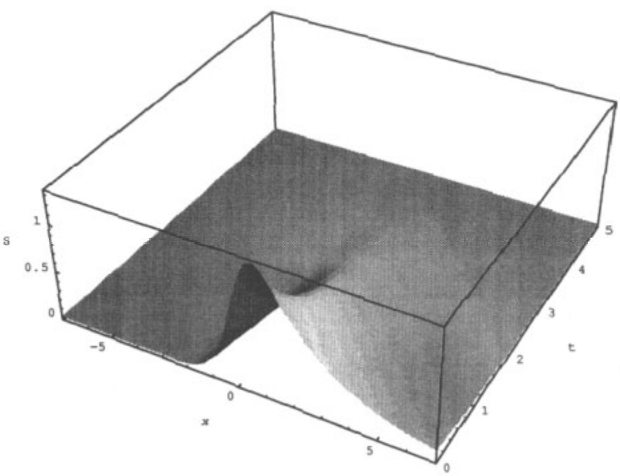

Figure 2. Solution of (16) for the case 2 (ii). The parameters are set to $k_2 = 0, k_1 = k_3 = m = 1$.

(ii) $f(x,t) = 0$, $g(x,t) = f_2(x)$. According to (34) and (35) f_2 has to solve the same ODE as f_1, so we can set $f_2 = f_1$. The solution $S(x,t)$ is almost the same as given in (38) except the '1' in the denominator. The integrated form is represented in

$$S(x,t) = k_3 \exp\left[-k_1 t + \int f_1(x)\right]. \tag{39}$$

For certain parameters a three-dimensional plot of $S(x,t)$ is given in figure 2.

Here we have found a different separation ansatz which cannot be obtained by Lie's original method.

5. Coupled KdV-type equations

The system of coupled KdV-type equations is another example to demonstrate the application of the non-classical method:

$$u_t + 3vv_x = 0 \tag{40}$$

$$v_t + 2v_{xxx} + 2uv_x + u_x v = 0\,. \tag{41}$$

Equations (40) and (41) can be found in the study of Kac–Moody algebras and in soliton theory [24, 25]. They give general properties for this type of equation in a KdV hierarchy, e.g. all conserved quantities can be written as differential polynomials in u and v.

Up to now the non-classical method has only been applied to single PDEs [18, 20, 28, 29]. All examples discussed in the literature are restricted to the case with one dependent variable. With our example, we will demonstrate that the non-classical method is also useful in the discussion of systems of PDEs.

If we use the classical symmetry reduction to examine the system of equations (40) and (41), we find a three-dimensional symmetry group represented by the infinitesimals

$$\xi^1 = c_1 - \frac{c_3}{2}x \tag{42}$$

$$\xi^2 = c_2 - \tfrac{3}{2}c_3 t \tag{43}$$

$$\phi^1 = c_3 u \tag{44}$$

$$\phi^2 = c_3 v\,. \tag{45}$$

Again ξ^1 and ξ^2 are the infinitesimals for the independent variables x and t. ϕ^1 and ϕ^2 are the infinitesimals related to u and v, respectively. The constants c_1, c_2, and c_3 are group constants characterizing the point symmetry of (40) and (41). From the infinitesimals it is obvious that (40) and (41) are invariant under space and time translations and possess a scaling invariance in the dependent and independent variables with the similarity variable $z = xt^{-1/3}$.

Similar to the equation discussed in section 4, we find that the symmetries of the non-classical method contains the symmetries of the classical method in some limiting cases. The application of the non-classical theory gives us an ansatz for the infinitesimals by

$$\xi^1 = a(t) + xb(t)$$

$$\xi^2 = 1$$

$$\phi^1 = c(t) + xd(t) - 2a(t)u$$

$$\phi^2 = g(t)v\,.$$

The functions $a(t), b(t), \ldots, g(t)$ solve an *overdetermined nonlinear* system of ODEs which can be solved by integration:

$$ab + 2ag - a' = 0 \tag{46}$$

$$2c - 3ab - a' = 0 \tag{47}$$

$$b^2 + 2bg - b' = 0 \tag{48}$$

$$-2d + 3b^2 + b' = 0 \tag{49}$$

$$-cb - 2cg + c' = 0 \tag{50}$$

$$-db - 2dg + d' = 0 \tag{51}$$

$$d + 3bg + g' = 0 \tag{52}$$

$$2ad + 2bc - 6ab^2 - ba' - ab' = 0 \tag{53}$$

$$-4cd + 3abd + 3cb^2 + da' + cb' = 0 . \tag{54}$$

The link between the classical and non-classical symmetries is made if we set $c(t) = d(t) = 0$ in (46)–(54) which leads to the classical result. Besides the classical limit, there are two different cases which we will take into consideration. First let us assume $b(t) = 0$.

Case 1: $b(t) = 0$. By this assumption the infinitesimals reduce to

$$\xi^1 = \frac{k_2}{k_1} e^{2k_1 t} \tag{55}$$

$$\xi^2 = 1 \tag{56}$$

$$\phi^1 = k_2 e^{2k_1 t} \tag{57}$$

$$\phi^2 = k_1 . \tag{58}$$

Solving (11) delivers the ansatz for $u(x, t)$ and $v(x, t)$

$$u(x,t) = w(z) + k_1 x \qquad v(x,t) = e^{k_1 t} f(z)$$

$$z = \frac{k_1}{k_2} x - \frac{1}{2k_1} e^{2k_1 t} .$$

Equations (40) and (41) reduce to a system of ODEs

$$-w'(z) + 3\frac{k_1}{k_2} f(z) f'(z) = 0 \tag{59}$$

$$2k_1 z f'(z) + k_1 f(z) + f(z)\left(w'(z)\frac{k_1}{k_2} + k_1\right) + 2\frac{k_1}{k_2} w(z) f'(z) + 2\left(\frac{k_1}{k_2}\right)^3 f'''(z) = 0 \tag{60}$$

which can be decoupled by integrating (59):

$$\frac{3k_1}{2k_2}(f(z))^2 + a = w(z) \tag{61}$$

$$f''(z) + \frac{k_2^3}{k_1^2} z f(z) + \frac{k_2}{k_1}(f(z))^3 + \frac{2ak_2^2}{k_1^2} f(z) + \frac{c_1 k_2^3}{2k_1^3} = 0 . \tag{62}$$

Equation (62) can be solved in terms of the second Painlevé transcendent [26] by the transformation

$$f(z) = \lambda \eta(\xi) \qquad \xi = \mu\left(\frac{k_2^3}{k_1^2} z + 2a\frac{k_2^2}{k_1^2}\right) \qquad \eta'' - 2\eta^3 - \xi\eta + \frac{c_1 k_1}{2k_2^3} = 0 . \tag{63}$$

Thus the solutions of $b(t) = 0$ correspond to second Painlevé transcendents. This is a completely new and different solution for $u(x, t)$ and $v(x, t)$ which is not scale invariant as is the solution obtainable with the classical method.

In our second case, we generalize the first one by assuming $b(t) \neq 0$.

Case 2: $b(t) \neq 0$. We take a particular solution for the infinitesimals as the surface condition cannot be solved for the complete solution

$$\xi^1 = x\left(-\tfrac{1}{3}k_1 \tanh(-k_1 t) + \tfrac{1}{3}k_1\right) \tag{64}$$

$$\xi^2 = 1 \tag{65}$$

$$\phi^1 = x\left(-\tfrac{1}{9}k_1^2 \tanh(-k_1 t) + \tfrac{1}{9}k_1^2\right) - 2\left(-\tfrac{1}{3}k_1 \tanh(-k_1 t) + \tfrac{1}{3}k_1\right)u \tag{66}$$

$$\phi^2 = .\left(\tfrac{2}{3}k_1 \tanh(-k_1 t) + \tfrac{1}{3}k_1\right)v . \tag{67}$$

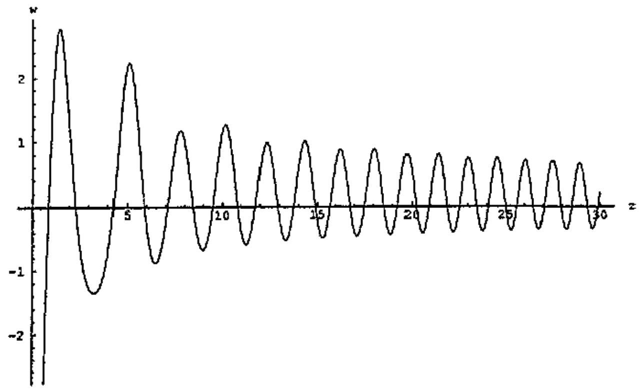

Figure 3. Numerical solution $w(z)$ of (71) and (72). The choice of the parameter is $k_1 = 1$ and the initial values are set to $f(1) = 1, f'(1) = 1, f''(1) = 0, w(1) = 1$.

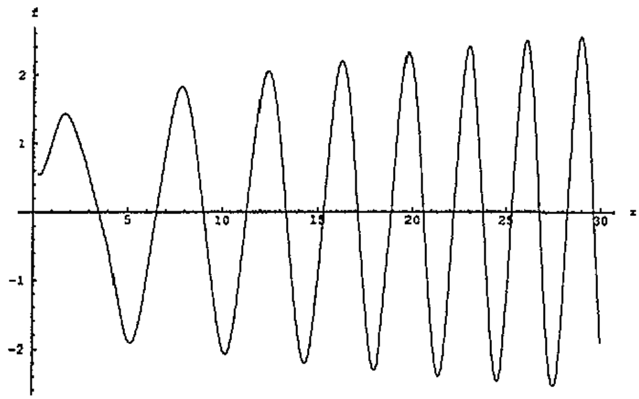

Figure 4. Numerical solution $f(z)$ of (71) and (72). The choice of the parameter is $k_1 = 1$ and the initial values are set to $f(1) = 1, f'(1) = 1, f''(1) = 0, w(1) = 1$.

So $u(x,t)$ and $v(x,t)$ can be expressed in terms of the invariants w, f and z by

$$u(x,t) = w(z)\frac{\mathrm{e}^{-2k_1t/3}}{[\cosh(-k_1t)]^{2/3}} + \frac{k_1}{3}x \tag{68}$$

$$v(x,t) = f(z)\frac{\mathrm{e}^{k_1t/3}}{[\cosh(-k_1t)]^{2/3}} \tag{69}$$

$$z = \frac{x\mathrm{e}^{k_1t/3}}{[\cosh(-k_1t)]^{1/3}}. \tag{70}$$

Inserting these expressions into (40) and (41), we obtain a coupled system of ODEs in $w(z)$ and $f(z)$

$$-\tfrac{1}{3}k_1zw'(z) - \tfrac{2}{3}k_1w(z) + 3f(z)f'(z) = 0 \tag{71}$$

$$\tfrac{2}{3}k_1f(z) + \tfrac{1}{3}k_1zf'(z) + 2f'''(z) - zf'(z)w'(z) + 9k_1f(z)(f'(z))^2 + w'(z)f(z) = 0. \tag{72}$$

This ansatz for $u(x,t)$ and $v(x,t)$ can neither be compared with case 1 nor with the results which can be derived by Lie's method of symmetry reduction. We get a completely new class of solutions.

This system is coupled in a non-trivial way and can only be solved numerically. A numerical solution obtained by MATHEMATICA is plotted for $k_1 = 1$ in figures 3 and 4.

6. Nonlinear wave equation

Finally we apply the non-classical method to a type of nonlinear wave equations $u_{tt} = (f(u)u_x)_x$. The group properties and the associated Lie algebra of this equation are developed in [11], also including physical examples which can be described by types of nonlinear wave equations. As an explicit example we consider the longitudinal wave propagation on a moving threadline. The governing equations of this model are

$$V_t + VV_x = \frac{\lambda}{\rho_0}\sigma_x \tag{73}$$

$$\lambda_t + V\lambda_x - \lambda V_x = 0 \tag{74}$$

$$\sigma = \sigma(\lambda) = E_0 r(\lambda) \tag{75}$$

where V, λ, σ and ρ_0 denote longitudinal velocity, stretch, stress and constant density, respectively. The system of equations (73) and (74) can be transformed to the equation we want to examine if we set

$$\psi_x = \frac{1}{\lambda} \qquad \psi_t = -\frac{V}{\lambda}.$$

Then λ has to satisfy a PDE in the independent variables t and ψ

$$\lambda_{tt} = \mu^2[r(\lambda)\lambda_\psi]_\psi \qquad \text{with} \quad \mu^2 = \frac{E_0}{\rho_0}. \tag{76}$$

If we assume for the stretch–stress relation Hooke's law

$$\sigma = E_0\lambda$$

we get an equation representing the case with $f(u) = u$

$$u_{tt} - (u_x)^2 - uu_{xx} = 0. \tag{77}$$

This equation has got a four-dimensional point symmetry group with the infinitesimals

$$\xi^1 = c_1 + c_2 x \tag{78}$$

$$\xi^2 = c_3 + c_4 t \tag{79}$$

$$\phi^1 = 2(c_2 - c_4)u. \tag{80}$$

As in the previous examples ξ^1, ξ^2 and ϕ^1 are related to x, t and u. The infinitesimals generate solutions which are invariant under space and time translations. Additionally, scaling invariant similarity solutions exist.

The non-classical method yields another different solution:

$$\xi^1 = k_1 t \qquad \xi^2 = 1 \qquad \phi^1 = 2k_1^2 t.$$

For simplicity we set $k_1 = 1$ and obtain

$$u = w(z) + t^2 \qquad \text{with} \quad z = x - \tfrac{1}{2}t^2. \tag{81}$$

Equation (77) reduces by using (81) to an ODE which can be easily integrated:

$$-ww' - w + 2z + a = 0 \tag{82}$$

where a is a constant of integration. The solution of (82) can be obtained after extensive analytical calculations by

$$w(z) = \frac{1}{2}a - z + \frac{3(a-2z)^2}{2^{2/3}\sqrt[3]{f(z)+g(z)}} + \frac{1}{6\sqrt[3]{2}}\sqrt[3]{f(z)+g(z)}$$

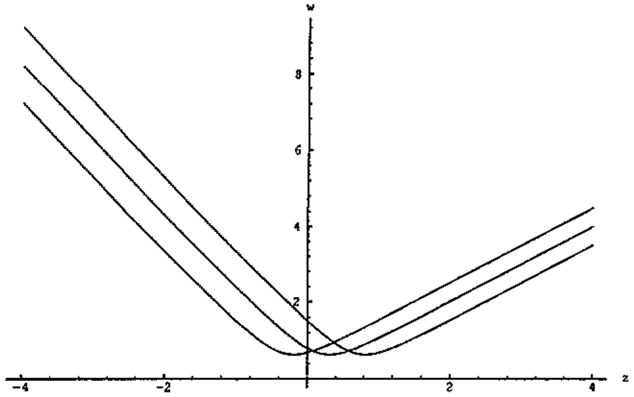

Figure 5. Solution $w(z)$ of the reduced nonlinear wave equation (82). The specific choice of the integration constants is $c_1 = 0, a = -1, 0, 1$. From left to right the curves correspond to $a = -1, a = 0, a = 1$.

with

$$f(z) = 54(a - 2z)^3 - 108(a^3 - e^{3c_1} - 6a^2z + 12az^2 - 8z^3)$$

being a polynomial in z of order 3 and

$$g(z) = \sqrt{-2916(a - 2z)^6 + f^2(z)}$$

being the square root of an extended polynomial of f. A graphical representation of $w(z)$ is plotted for $c_1 = 0, a = -1, a = 0, a = 1$ in figure 5.

In this section we have found a new solution of the nonlinear wave equation (77) for a generalized 'travelling wave' variable which is essentially different from the solutions calculated in [11] which can be obtained by the infinitesimals given in (78)–(80). In the article by Ames and Lohner [11] solutions with scaling variables are derived from (77) in the form

$$u(x, t) = (t + k_1)^{k_2} g(z) \qquad \text{with} \quad z = \frac{x + k_3}{(t + k_4)^{k_5}}$$

where $g(z)$ has to satisfy a nonlinear ODE of second order in z

$$ag(z) + bzg'(z) + cz^2g''(z) = d(gg'(z))' .$$

The constants a, b, c and d depend on the former constants k_i. Ames and Lohner demonstrated in [11] that singularities for a certain choice of parameters can occur for $u(x, t)$. In contrast to this behaviour, our solution $w(z)$ is defined for all real-valued z and is linear in z with fixed slope 1 in the asymptotic case $|z| \gg 1$ independent of any group parameters.

7. Conclusions

We have demonstrated that the non-classical method is a useful and effective tool to discover new solutions of PDEs. These solutions are different from those which can be obtained by Lie's method of symmetry reduction. The results derived by both methods are summarized in table 1. A disadvantage of both the non-classical method and the classical method of Lie is the absence of special initial and boundary values. In the case of the classical method,

Table 1. Comparison between the classical and the non-classical method. For the non-classical method several different solutions of the infinitesimals are listed. The function $f(x,t)$ has to satisfy equation (20) and the functions $f(x)$ and $f_1(x)$ are given in (28) and (37), respectively.

PDE	Classical method	Non-classical method
	$\xi^1 = c_2 + c_3 t - \frac{c_3}{2m}x$	$\xi^1 = f(x)$
	$+\frac{c_5}{m}x + 4c_6 tx$	$\xi^1 = 1, \xi^1 = 1$
$SS_t - SS_{xx} + 2S_x^2$	$\xi^2 = c_1 - \frac{c_3}{m}t + \frac{2c_5}{m}t + 4c_6 t^2$	$\xi^2 = 1, \xi^2 = 0, \xi^2 = 0$
$-2mSS_x = 0$	$\phi^1 = (c_4 + 2c_5 mt + c_5 x)S$	$\phi^1 = 0,$
	$+(2c_6 t + 4c_6 m^2 t^2 + c_6 x^2$	$\phi^1 = f_1(x)(S + S^2),$
	$+4c_6 mtx)S + f(x,t)S^2$	$\phi^1 = f_1(x)S$
	$\xi^1 = c_1 - \frac{1}{2}c_3 x$	$\xi^1 = \frac{k_2}{k_1}e^{2k_1 t}, \xi^1 = \frac{1}{3}xk_1(1 - \tanh(-k_1 t))$
$u_t + 3vv_x = 0$	$\xi^2 = c_2 - \frac{3}{2}c_3 t$	$\xi^2 = 1, \xi^2 = 1$
$v_t + vu_x + 2uv_x$	$\phi^1 = c_3 u$	$\phi^1 = k_2 e^{2k_1 t},$
$+2v_{xxx} = 0$		$\phi^1 = \frac{1}{9}xk_1^2(1 - \tanh(-k_1 t)) - \frac{2}{3}k_1(1 - \tanh(-k_1 t))u$
	$\phi^2 = c_3 v$	$\phi^2 = k_1, \phi^2 = \frac{1}{3}k_1(2\tanh(-k_1 t) + 1)v$
	$\xi^1 = c_1 + c_2 x$	$\xi^1 = k_1 t$
$u_{tt} - (uu_x)_x = 0$	$\xi^2 = c_3 + c_4 t$	$\xi^2 = 1$
	$\phi^1 = 2(c_2 - c_4)u$	$\phi^1 = 2k_1^2 t$

boundary value problems are taken into consideration in [4, 10]. However, for nonlinear PDEs too many restrictions are involved.

With our calculations we have demonstrated that the non-classical method can deliver an ansatz to separate independent variables. This method is very useful if we remember that there is more than one possibility for a separation. Another useful property of the non-classical method of symmetry reduction is its applicability to systems of PDEs. Up to now the procedure of the non-classical method of symmetry reduction has been applied to scalar PDEs, i.e. to cases with one dependent variable. With our calculations using the non-classical method for a system of PDEs we have derived new classes of solutions. The resulting equations cannot be decoupled in their most general cases. We note that the reduction of the equations is possible with the non-classical method. One of these reductions results in the second Painlevé transcendent, which is known to be solvable.

Another way of finding new classes of solutions for PDEs is the 'direct method' introduced by Clarkson and Kruskal [27]. It has been shown that the non-classical method is more general than the direct method [28, 29]. The reductions in part of the examples treated in this paper, i.e. whenever a similarity variable occurs, should be obtainable with the direct method as well [29].

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